# PRIMITIVE RADON PARTITIONS FOR CYCLIC POLYTOPES

BY

MARILYN BREEN

ABSTRACT

It is proved that the set $A \cup B$ is a primitive Radon partition in the cyclic polytope $C(n,d)$ if and only if the $d+2$ points of $A$ and $B$ alternate along the moment curve.

The reader is referred to [3] and [1] for a discussion of primitive partitions. We begin with a few preliminary remarks concerning cyclic polytopes.

Let $M_d$ denote the moment curve in $R^d$ defined by $\{x(t) = (t, t^2, \cdots, t^d)\colon t \text{ real}\}$. Let $t_1 < t_2 < \cdots < t_n$ be any $n$ distinct values of the parameter $t$, $n \geqq d+1$, and let $V \equiv \{x(t_i)\colon i = 1, 2, \cdots, n\}$. Then the polytope $\operatorname{conv} V \equiv C(n,d)$ is said to be a *cyclic d-polytope*. It is known [2] that every cyclic polytope is $[d/2]$-neighborly with vertices in general position in $R^d$, and that every two cyclic polytopes $C(n,d)$ are combinatorially equivalent.

THEOREM. *The set $A \cup B$ is a primitive Radon partition in the cyclic polytope $C(n,d)$ if and only if the $d+2$ points of $A$ and $B$ alternate along the moment curve.*

PROOF. Since $C(n,d)$ has its vertices in general position in $R^d$, each primitive partition $A \cup B$ for $C(n,d)$ contains exactly $d+2$ points. Furthermore, since $C(n,d)$ is $[d/2]$-neighborly, by [1, Th. 4], for $j \leqq [d/2]$, no $j$-member subset of $V$ is half a primitive in $V$. Thus, one of $A$, $B$ has cardinality $[d/2]+1$. If card $B = [d/2]+1$, then card $A = d - [d/2] + 1$, and card $A$ − card $B$ is either 0 or 1.

Let $S$ denote the $(d+2)$-member subset of $V$ containing $A$ and $B$, where

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$S = \{x(s_i): 1 \leqq i \leqq d+2\}$ is labeled so that $s_1 < s_2 < \cdots < s_{d+2}$. Then $S$ is in general position in $R^d$, and the primitive partition $A \cup B$ for $S$ is unique.

Examine the cyclic polytope conv $S$. By [1, Th. 4, Corollary 1], no superset of $A$ (or of $B$) is a facet of conv $S$, so each facet of conv $S$ omits a member of $A$ and a member of $B$. Moreover, since each facet is determined by a $d$-member subset of $S$, and card $S = d+2$, each facet of conv $S$ omits exactly one member $x(a_0)$ in $A$ and exactly one $x(b_0)$ in $B$. By Gale's evenness condition [2, Th. 4.7.2], this is true if and only if $a_0$ and $b_0$ are separated in our ordering by an even number of the remaining $s_i$ points.

Also, every pair $\{a,b\}$, $x(a) \in A$, $x(b) \in B$, determines a facet of conv $S$, since no subset of $S \sim \{x(a), x(b)\}$ is half a primitive for $S$. Therefore, for every $a$ and $b$, $a$ and $b$ are separated in our ordering by an even number of the remaining $s_i$ points. Without loss of generality, assume $x(s_1) \in A$. Then

$$A = \{x(s_i): i \text{ odd}, 1 \leqq i \leqq d+2\},$$

$$B = \{x(s_i): i \text{ even}, 1 \leqq i \leqq d+2\},$$

and the points of $A$ and $B$ alternate on $S$.

Conversely, let $A$ and $B$ be disjoint subsets of $V$, card $A$ + card $B = d+2$, and assume that the points of $A$ and $B$ alternate on the subset $S$ of $M_d$ containing them. There is a unique partition for $S$, and by applying the above argument, this partition must be $A \cup B$. Thus the Radon partitions for $C(n,d)$ are uniquely determined by $M_d$, completing the proof.

## References

1. Marilyn Breen, *Determining a polytope by Radon partitions*, Pacific J. Math., **43** (1972), 27–37.

2. Branko Grünbaum, *Convex Polytopes*, New York, 1967.

3. William R. Hare and John W. Kenelly, *Characterizations of Radon partitions*, Pacific J. Math. **36** (1971), 159–164.

University of Oklahoma
Norman, Oklahoma, U. S. A. 73609